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Spectral Flows and Twisted Topological Theories

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ABSTRACT

We analyze the action of the spectral flows on N=2 twisted topological theories. We show that they provide a useful mapping between the two twisted topological theories associated to a given N=2 superconformal theory. This mapping can also be viewed as a topological algebra automorphism. In particular null vectors are mapped into null vectors, considerably simplifying their computation. We give the level 2 results. Finally we discuss the spectral flow mapping in the case of the DDK and KM realizations of the topological algebra.

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1 Introduction

We investigate the spectral flow transformations associated to N=2 superconformal theories, when they are applied to the topological theories obtained by twisting the superconformal ones. Only spectral flows with $\theta = \pm 1$ provide a mapping between the two twisted topological theories associated to a given superconformal theory; this is due to the fact that only for those values the Hilbert spaces of the two twisted theories are mapped into each other. Although for no value of θ (except the trivial) the theories map back to themselves, the mapping interpolating between the two twisted theories gives rise to a topological algebra automorphism acting inside a given theory. In both cases we find that null vectors are mapped into null vectors, either from one twisted theory to the other or inside one of the theories. As a result the computation of null vectors reduces considerably, not only because there are less vectors in number to compute but also because hard to obtain null vectors are directly related to much easier ones. In section 3 we write down all the relevant expressions for level 2; that is, the different kinds of descendants and null vectors, and the spectral flow relations between them. In section 4 we study the spectral flow mappings for the DDK and KM realizations of the topological algebra, which are the two twistings of the same N=2 superconformal theory. The DDK realization is a bosonic string construction (with the Liouville field and $c = -26$ reparametrization ghosts), while the KM realization is related to the KP hierarchy through the Kontsevich-Miwa transformation. We obtain the spectral flow mapping relating the different field components between these two theories, finding that it does not mix fields of different nature. As a result the null vectors of the reduced conformal field theories (matter + scalar systems, without the ghosts) are mapped into each other as well. Section 5 is devoted to conclusions.

2 Spectral Flow Mappings

The topological algebra obtained by twisting the N=2 superconformal algebra [1], [2], [3] reads

$$\begin{array}{lcllcl}
[\mathcal{L}_m, \mathcal{L}_n] & = & (m-n)\mathcal{L}_{m+n}\,, & [\mathcal{H}_m, \mathcal{H}_n] & = & \frac{\mathsf{c}}{3} m \delta_{m+n,0}\,, \\
{[\mathcal{L}_m, \mathcal{G}_n]} & = & (m-n)\mathcal{G}_{m+n}\,, & [\mathcal{H}_m, \mathcal{G}_n] & = & \mathcal{G}_{m+n}\,, \\
{[\mathcal{L}_m, \mathcal{Q}_n]} & = & -n\mathcal{Q}_{m+n}\,, & [\mathcal{H}_m, \mathcal{Q}_n] & = & -\mathcal{Q}_{m+n}\,, \qquad m,\ n \in \mathbf{Z}\,. \\
{[\mathcal{L}_m, \mathcal{H}_n]} & = & -n\mathcal{H}_{m+n} + \frac{\mathsf{c}}{6}(m^2+m)\delta_{m+n,0}\,, & & & \\
\{\mathcal{G}_m, \mathcal{Q}_n\} & = & 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{\mathsf{c}}{3}(m^2+m)\delta_{m+n,0}\,, & & &
\end{array} \tag{2.1}$$

where $\mathcal{L}_m$ and $\mathcal{H}_m$ are the bosonic generators corresponding to the energy momentum

tensor (Virasoro generators) and the topological $U(1)$ current respectively, while $\mathcal{Q}_m$ and $\mathcal{G}_m$ are the fermionic generators corresponding to the BRST current and the spin-2 fermionic current respectively. The "topological central charge" $\mathtt{c}$ is the true central charge of the N=2 superconformal algebra [5] .

This algebra is satisfied by the two sets of topological generators

$$
\begin{aligned}
\mathcal{L}_m^{(1)} &= L_m + \tfrac{1}{2}(m+1)H_m\,, \\
\mathcal{H}_m^{(1)} &= H_m\,, \\
\mathcal{G}_m^{(1)} &= G^+_{m+\frac{1}{2}}\,, \qquad \mathcal{Q}_m^{(1)} = G^-_{m-\frac{1}{2}}\,,
\end{aligned}
\tag{2.2}
$$

and

$$
\begin{aligned}
\mathcal{L}_m^{(2)} &= L_m - \tfrac{1}{2}(m+1)H_m\,, \\
\mathcal{H}_m^{(2)} &= -H_m\,, \\
\mathcal{G}_m^{(2)} &= G^-_{m+\frac{1}{2}}\,, \qquad \mathcal{Q}_m^{(2)} = G^+_{m-\frac{1}{2}}\,,
\end{aligned}
\tag{2.3}
$$

corresponding to the two possible twistings of the superconformal generators L_m, H_m, G^+_m and G^-_m. We see that G^+ and G^- play mirrored roles with respect to the definitions of $\mathcal{G}$ and $\mathcal{Q}$. In particular $(G^+_{1/2}, G^-_{-1/2})$ results in $(\mathcal{G}_0^{(1)}, \mathcal{Q}_0^{(1)})$, while $(G^-_{1/2}, G^+_{-1/2})$ gives $(\mathcal{G}_0^{(2)}, \mathcal{Q}_0^{(2)})$, so that the topological primary chiral fields $\Phi^{(1)}$ and $\Phi^{(2)}$ come from the antichiral and the chiral rings respectively.

The spectral flows act on the superconformal generators in the following way [4], [5]

$$
\begin{aligned}
\mathcal{U}_\theta\, L_m\, \mathcal{U}_\theta^{-1} &= L_m + \theta H_m + \tfrac{\mathtt{c}}{6}\theta^2\delta_{m,0}\,, \\
\mathcal{U}_\theta\, H_m\, \mathcal{U}_\theta^{-1} &= H_m + \tfrac{\mathtt{c}}{3}\theta\delta_{m,0}\,, \\
\mathcal{U}_\theta\, G_r^+\, \mathcal{U}_\theta^{-1} &= G^+_{r+\theta}\,, \\
\mathcal{U}_\theta\, G_r^-\, \mathcal{U}_\theta^{-1} &= G^-_{r-\theta}\,.
\end{aligned}
\tag{2.4}
$$

This translates at the level of the topological generators (2.2) and (2.3) into the transformations

$$
\begin{aligned}
\mathcal{U}_\theta\, \mathcal{L}_m^{(2)}\, \mathcal{U}_\theta^{-1} &= \mathcal{L}_m^{(1)} + (\theta-(m+1))\mathcal{H}_m^{(1)} + \tfrac{\mathtt{c}}{6}\theta(\theta-(m+1))\delta_{m,0}\,, \\
\mathcal{U}_\theta\, \mathcal{H}_m^{(2)}\, \mathcal{U}_\theta^{-1} &= -\mathcal{H}_m^{(1)} - \tfrac{\mathtt{c}}{3}\theta\delta_{m,0}\,, \\
\mathcal{U}_\theta\, \mathcal{Q}_m^{(2)}\, \mathcal{U}_\theta^{-1} &= \mathcal{G}^{(1)}_{m-1+\theta}\,, \\
\mathcal{U}_\theta\, \mathcal{G}_m^{(2)}\, \mathcal{U}_\theta^{-1} &= \mathcal{Q}^{(1)}_{m+1-\theta}\,,
\end{aligned}
\tag{2.5}
$$

and exactly the same expressions exchanging $(1) \leftrightarrow (2)$ and $\theta \leftrightarrow -\theta$. We could have expressed the right hand side of these transformations in terms of the generators (2) instead (with different expressions, of course). However, this is not of much use, as we

will now discuss. First of all, it is clear that θ must be an integer for the transformation to make any sense. Moreover, θ must be equal to 1. The reason is that only $\mathcal{U}_1$ maps the chiral ring, on which the generators (2) act, into the antichiral ring, on which the generators (1) act. In addition, for no value of θ (except the trivial $\theta = 0$) the chiral ring gets mapped into the chiral ring, making it completely useless to express the right hand side of (2.5) in terms of the generators (2) (because only the fields $\Phi^{(2)}$ are chiral primaries with respect to these). Therefore, the only sensible spectral flow mapping for the topological generators (2) is given by

$$\begin{aligned} \mathcal{U}_1\, \mathcal{L}^{(2)}_m\, \mathcal{U}_1^{-1} &= \mathcal{L}^{(1)}_m - m\mathcal{H}^{(1)}_m \,, \\ \mathcal{U}_1\, \mathcal{H}^{(2)}_m\, \mathcal{U}_1^{-1} &= -\mathcal{H}^{(1)}_m - \tfrac{\mathsf{c}}{3}\delta_{m,0} \,, \\ \mathcal{U}_1\, \mathcal{Q}^{(2)}_m\, \mathcal{U}_1^{-1} &= \mathcal{G}^{(1)}_m \,, \\ \mathcal{U}_1\, \mathcal{G}^{(2)}_m\, \mathcal{U}_1^{-1} &= \mathcal{Q}^{(1)}_m \,. \end{aligned} \tag{2.6}$$

For $\theta = -1$ the antichiral ring maps into the chiral ring, and one gets the same transformation (2.6) exchanging $(1) \leftrightarrow (2)$ and $\mathcal{U}_1 \leftrightarrow \mathcal{U}_{-1}$.

Now let us consider a descendant $|\Upsilon\rangle^{(2)}$ of the twisted theory (2) and its image under $\mathcal{U}_1$, that is a descendant of the twisted theory (1), say $|\Upsilon\rangle^{(1)} = \mathcal{U}_1 |\Upsilon\rangle^{(2)}$. From (2.6) we get

$$\begin{aligned} \mathcal{L}^{(1)}_0\, |\Upsilon\rangle^{(1)} &= \mathcal{U}_1\, \mathcal{L}^{(2)}_0\, |\Upsilon\rangle^{(2)} = \Delta^{(2)}\, |\Upsilon\rangle^{(1)} \,, \\ -(\mathcal{H}^{(1)}_0 + \tfrac{\mathsf{c}}{3})\, |\Upsilon\rangle^{(1)} &= \mathcal{U}_1\, \mathcal{H}^{(2)}_0\, |\Upsilon\rangle^{(2)} = \mathsf{h}^{(2)}\, |\Upsilon\rangle^{(1)} \,, \\ \mathcal{Q}^{(1)}_0\, |\Upsilon\rangle^{(1)} &= \mathcal{U}_1\, \mathcal{G}^{(2)}_0\, |\Upsilon\rangle^{(2)} \,, \\ \mathcal{G}^{(1)}_0\, |\Upsilon\rangle^{(1)} &= \mathcal{U}_1\, \mathcal{Q}^{(2)}_0\, |\Upsilon\rangle^{(2)} \,. \end{aligned} \tag{2.7}$$

That is, the spectral flow $\mathcal{U}_1$ leaves the conformal weight (the level) of the descendant $|\Upsilon\rangle^{(2)}$ unchanged, $\Delta^{(1)} = \Delta^{(2)}$, while modifying the U(1) charge, $\mathsf{h}^{(1)} = -\mathsf{h}^{(2)} - \mathsf{c}/3$. Moreover, $\mathcal{Q}^{(2)}_0$- invariant ($\mathcal{G}^{(2)}_0$-invariant) descendants $|\Upsilon\rangle^{(2)}$ transform into $\mathcal{G}^{(1)}_0$-invariant ($\mathcal{Q}^{(1)}_0$-invariant) descendants $|\Upsilon\rangle^{(1)}$.

In addition, if $|\Upsilon\rangle^{(2)}$ is a null vector, the highest weight conditions $\mathcal{L}^{(2)}_{m>0}|\Upsilon\rangle^{(2)} = \mathcal{H}^{(2)}_{m>0}|\Upsilon\rangle^{(2)} = \mathcal{G}^{(2)}_{m>0}|\Upsilon\rangle^{(2)} = \mathcal{Q}^{(2)}_{m>0}|\Upsilon\rangle^{(2)} = 0$ result in highest weight conditions for $|\Upsilon\rangle^{(1)}$ too. Conversely, if $|\Upsilon\rangle^{(1)}$ is a null vector, $\mathcal{U}_{-1}|\Upsilon\rangle^{(1)} = |\Upsilon\rangle^{(2)}$ is a null vector as well. Notice that the spectral flow (2.4) acting on the untwisted N=2 superconformal algebra does not map null vectors built on the chiral ring into null vectors built on the antichiral ring. It even changes the level of the descendants, depending on their $U(1)$ charges. The so called mirror map between the two twisted topological theories [6] also fails to map null states into null states.

At this point we have to make an observation. In dealing with topological descendants, constructing null vectors, etc, as long as we stay at the topological algebra level, without going into particular realizations, we can regard the transformation (2.6) (without the labels (1) and (2)) as an internal mapping of the topological algebra (2.1). As a matter

of fact, that transformation without the labels is *an automorphism of the topological algebra*, as the reader can easily verify. This is a reflection of the fact that for the N=2 superconformal algebra the spectral flows for $\theta = \pm 1$ commute with the twistings. In order to be rigorous we must trade the spectral flow operator $\mathcal{U}_1$ by an operator, say $\mathcal{A}$, such that the topological algebra automorphism can be properly expressed as

$$\begin{aligned} \mathcal{A}\,\mathcal{L}_m\,\mathcal{A}^{-1} &= \mathcal{L}_m - m\mathcal{H}_m\,, \\ \mathcal{A}\,\mathcal{H}_m\,\mathcal{A}^{-1} &= -\mathcal{H}_m - \tfrac{\mathsf{c}}{3}\delta_{m,0}\,, \\ \mathcal{A}\,\mathcal{Q}_m\,\mathcal{A}^{-1} &= \mathcal{G}_m\,, \\ \mathcal{A}\,\mathcal{G}_m\,\mathcal{A}^{-1} &= \mathcal{Q}_m\,. \end{aligned} \tag{2.8}$$

It is straightforward to prove that $\mathcal{A}^{-1} = \mathcal{A}$ and that $\mathcal{A}$, like $\mathcal{U}_1$, maps null vectors into null vectors of the same level and U(1) charges related by $\mathsf{h} \leftrightarrow -\mathsf{h} - \mathsf{c}/3$. Again, this has no direct analog neither in the N=2 superconformal case nor for the mirror automorphism of the twisted topological algebra. In fact, we have found the operators that transform null states built on the chiral ring into both null states built on the chiral ring and null states built on the antichiral ring; they will be published in a forthcoming paper [7].

3 Level 2 Results

Now we will apply the results obtained in the previous section to the case of level 2 descendants and null vectors. We will adopt the *algebra automorphism* point of view, and use the transformation (2.8). Equivalently, we could have used the spectral flow mapping (2.6) between the two twisted theories, keeping track of the labels (1) and (2).

Let us start by noticing that any topological descendant $|\Upsilon\rangle$ can be decomposed as the sum of a $\mathcal{Q}_0$-invariant descendant $|\Upsilon\rangle^Q$ and a $\mathcal{G}_0$-invariant descendant $|\Upsilon\rangle^G$, as deduced from the anticommutator $\{\mathcal{Q}_0, \mathcal{G}_0\} = 2\mathcal{L}_0$. For this reason we will concentrate mainly on those states.

Let us call $\mathcal{O}^{(q)}$ the operator acting on the primary state $|\Phi\rangle_{\mathsf{h}}$, with U(1) charge h, to build the descendant $|\Upsilon\rangle^{(q)}$, namely $|\Upsilon\rangle^{(q)} = \mathcal{O}^{(q)}|\Phi\rangle_{\mathsf{h}}$. The *relative* U(1) charge q of $|\Upsilon\rangle^{(q)}$ is the charge carried by $\mathcal{O}^{(q)}$, given by the number of $\mathcal{G}$ modes minus the number of $\mathcal{Q}$ modes in each term. The U(1) charge of $|\Upsilon\rangle^{(q)}$ is therefore $\mathsf{h}+q$. We will denote by $\mathcal{O}^{(q)Q}$ and $\mathcal{O}^{(q)G}$ the operators corresponding to $\mathcal{Q}_0$-invariant and $\mathcal{G}_0$-invariant descendants, $|\Upsilon\rangle^{(q)Q}$ and $|\Upsilon\rangle^{(q)G}$, respectively. The transformation (2.8) changes the U(1) charges of the descendants and primary states in the form $\mathsf{h} \leftrightarrow -\mathsf{h} - \frac{\mathsf{c}}{3}$. As a consequence the relative charges are mapped as $q \leftrightarrow -q$ and the operators of type $\mathcal{O}^{(-q)Q}$ are transformed into operators of type $\mathcal{O}^{(q)G}$.

At level 2 there are five kinds of operators $\mathcal{O}^{(q)}$ regarding the possible values of q

and the invariance properties under $\mathcal{Q}_0$ and $\mathcal{G}_0$ of the corresponding descendants $|\Upsilon\rangle^{(q)}$. These operators are $\mathcal{O}^{(-1)Q}, \mathcal{O}^{(1)G}, \mathcal{O}^{(0)}, \mathcal{O}^{(0)Q}$ and $\mathcal{O}^{(0)G}$. Notice that only for $q = 0$ there exist descendants that are neither $\mathcal{Q}_0$ nor $\mathcal{G}_0$ invariant (although a sum of both kinds, as we mentioned before). The general rule, at any level, is that the descendants with largest absolute value of q are either $\mathcal{Q}_0$-invariant (for $q < 0$) or $\mathcal{G}_0$-invariant (for $q > 0$).

The generic $\mathcal{O}^{(-1)Q}$ operator can be written as

$$\mathcal{O}^{(-1)Q} = \mathcal{Q}_{-2} + \alpha\mathcal{L}_{-1}\mathcal{Q}_{-1} + \beta\mathcal{H}_{-1}\mathcal{Q}_{-1} \tag{3.1}$$

The transformed operator reads

$$\mathcal{A}\mathcal{O}^{(-1)Q}\mathcal{A} = \mathcal{G}_{-2} + \alpha\mathcal{L}_{-1}\mathcal{G}_{-1} + (\alpha - \beta)\mathcal{H}_{-1}\mathcal{G}_{-1} \tag{3.2}$$

and is a generic $\mathcal{O}^{(1)G}$ operator. Null vectors of type $|\Upsilon\rangle^{(-1)Q}$ and $|\Upsilon\rangle^{(1)G} = \mathcal{A}|\Upsilon\rangle^{(-1)Q}$ are given by the two sets of solutions

$$\alpha = \begin{cases} \frac{6}{\mathsf{c}-3} \\ \frac{\mathsf{c}-3}{6} \end{cases}, \qquad \beta = \begin{cases} \frac{12}{\mathsf{c}-3} \\ \frac{\mathsf{c}+3}{6} \end{cases}, \qquad \alpha - \beta = \begin{cases} \frac{6}{3-\mathsf{c}} \\ -1 \end{cases}, \tag{3.3}$$

corresponding to the following U(1) charges of the primary states on which they are built

$$\mathsf{h}^Q = \begin{cases} \frac{1-\mathsf{c}}{2} \\ -\frac{\mathsf{c}+3}{2} \end{cases}, \qquad \mathsf{h}^G = \begin{cases} \frac{\mathsf{c}-3}{6} \\ 1 \end{cases}. \tag{3.4}$$

Notice that $\mathsf{h}^G = -\mathsf{h}^Q - \mathsf{c}/3$.

Now let us consider the operators with relative charge $q = 0$. The general expression for $\mathcal{O}^{(0)}$ is

$$\mathcal{O}^{(0)} = \alpha\mathcal{L}_{-1}^2 + \theta\mathcal{L}_{-2} + \Gamma\mathcal{H}_{-1}\mathcal{L}_{-1} + \beta\mathcal{H}_{-1}^2 + \gamma\mathcal{H}_{-2} + \delta\mathcal{Q}_{-1}\mathcal{G}_{-1} \tag{3.5}$$

$\mathcal{Q}_0$-invariance of the associated descendant gives the constraints

$$\beta = 0, \quad \gamma = 0, \quad \Gamma = 2\delta \tag{3.6}$$

while $\mathcal{G}_0$-invariance results in the equations

$$2\alpha - \Gamma + 2\delta = 0, \quad \alpha - \Gamma + \beta = 0, \quad 2\theta + \alpha - \gamma + 2\delta = 0 \tag{3.7}$$

The transformed operator is also of generic $\mathcal{O}^{(0)}$ type and reads

$$\begin{aligned}\mathcal{A}\mathcal{O}^{(0)}\mathcal{A} = \alpha\mathcal{L}_{-1}^2 \quad &+ \quad (2\delta+\theta)\mathcal{L}_{-2} + (2\alpha-\Gamma)\mathcal{H}_{-1}\mathcal{L}_{-1} + (\alpha-\Gamma+\beta)\mathcal{H}_{-1}^2 \\ &+ \quad (2\theta+\alpha-\gamma+2\delta)\mathcal{H}_{-2} - \delta\mathcal{Q}_{-1}\mathcal{G}_{-1}.\end{aligned} \tag{3.8}$$

It is straightforward to verify that this operator corresponds to a $\mathcal{G}_0$-invariant ($\mathcal{Q}_0$-invariant) descendant if the former operator corresponds to a $\mathcal{Q}_0$-invariant ($\mathcal{G}_0$-invariant) descendant.

For $\mathcal{Q}_0$-invariant null vectors $|\Upsilon\rangle^{(0)Q}$, using (3.6) and setting $\theta = 1$, one finds [8]

$$\mathcal{O}^{(0)Q} = \alpha\mathcal{L}_{-1}^2 + \mathcal{L}_{-2} + \Gamma\mathcal{H}_{-1}\mathcal{L}_{-1} + \frac{1}{2}\Gamma\mathcal{Q}_{-1}\mathcal{G}_{-1}\,, \tag{3.9}$$

with

$$\alpha = \begin{cases} \frac{6}{\mathsf{c}-3} \\ \frac{\mathsf{c}-3}{6} \end{cases}, \qquad \Gamma = \begin{cases} \frac{6}{3-\mathsf{c}} \\ -1 \end{cases}, \qquad \mathsf{h}^Q = \begin{cases} \frac{\mathsf{c}-3}{6} \\ 1 \end{cases}. \tag{3.10}$$

The transformed null vectors $|\Upsilon\rangle^{(0)G}$ given by the operator (3.8), are $\mathcal{G}_0$-invariant with $\mathsf{h}^G = -\mathsf{h}^Q - \mathsf{c}/3$. The resulting $\mathcal{O}^{(0)G}$ is given by (using (3.6) and (3.10))

$$2\delta+\theta = \begin{cases} \frac{9-\mathsf{c}}{3-\mathsf{c}} \\ 0 \end{cases}, \qquad 2\alpha-\Gamma = \begin{cases} \frac{18}{\mathsf{c}-3} \\ \frac{\mathsf{c}}{3} \end{cases}. \tag{3.11}$$

$$\alpha-\Gamma = \begin{cases} \frac{12}{\mathsf{c}-3} \\ \frac{\mathsf{c}+3}{6} \end{cases}, \qquad 2\theta+\alpha+2\delta = \begin{cases} 2 \\ \frac{\mathsf{c}+3}{6} \end{cases}, \qquad \mathsf{h}^G = \begin{cases} \frac{1-\mathsf{c}}{2} \\ -\frac{\mathsf{c}+3}{3} \end{cases}. \tag{3.12}$$

Observe that $\mathcal{O}^{(0)Q}$ is much simpler (four terms) than $\mathcal{O}^{(0)G}$ (six terms). The reason is that terms containing $\mathcal{H}$ modes alone are absent in $\mathcal{Q}_0$-invariant descendants of type $|\Upsilon\rangle^{(0)}$ [9]. This is true at any level, so that at any level the direct computation of $\mathcal{G}_0$-invariant null states $|\Upsilon\rangle^{(0)G}$ is much harder than that of $\mathcal{Q}_0$-invariant null states $|\Upsilon\rangle^{(0)Q}$ (already at level 2 the difference in computing time for them is one order of magnitude: roughly two hours for $|\Upsilon\rangle^{(0)Q}$ and about 20 hours for $|\Upsilon\rangle^{(0)G}$). By using the transformation (2.8), however, we obtain straightforwardly null states of type $|\Upsilon\rangle^{(0)G}$ from null states of type $|\Upsilon\rangle^{(0)Q}$ (in the example above the 20 hour work reduces to 15 minute work).

4 DDK and KM Realizations of the Topological Algebra

In this section we will use the spectral flow mapping (2.6) to relate the DDK and KM realizations of the topological algebra (2.1) [8] [9] [10]. These are the two twistings of the same N=2 superconformal theory . The DDK topological field theory, a bosonic string construction [11], corresponds to the twist (2), while the KM topological field theory, that is related to the KP hierarchy through the Kontsevich-Miwa transformation, corresponds to the twist (1). The field content in these realizations consists of $d \leq 1$ matter, a scalar and a bc system. The scalars differ by the background charge and the way they dress the matter ($Q_s = Q_{Liouville},\ \Delta = 1$ in the DDK case versus $Q_s = Q_{matter},\ \Delta = 0$ in the KM case), while the bc systems differ by the spin and the central charge ($s = 2,\ c = -26$ for the DDK ghosts versus $s = 1,\ c = -2$ for the KM ghosts).

In the DDK realization the generators of the topological algebra are

$$\mathcal{L}_m^{(2)} = L_m^{(2)} + l_m^{(2)}, \qquad l_m^{(2)} \equiv \sum_{n\in\mathbf{Z}} (m+n) :b_{m-n}^{(2)} c_n^{(2)}: \tag{4.1}$$

$$\mathcal{H}_m^{(2)} = \sum_{n\in\mathbf{Z}} :b_{m-n}^{(2)} c_n^{(2)}: - \sqrt{\frac{3-\mathsf{c}}{3}} I_m^{(2)}, \qquad \mathcal{G}_m^{(2)} = b_m^{(2)}\,, \tag{4.2}$$

$$\mathcal{Q}_m^{(2)} = 2\sum_{p\in\mathbf{Z}} c_{m-p}^{(2)} L_p^{(2)} + \sum_{p,r\in\mathbf{Z}} (p-r) :b_{m-p-r}^{(2)} c_p^{(2)} c_r^{(2)}: + 2\sqrt{\frac{3-\mathsf{c}}{3}}\, m \sum_{p\in\mathbf{Z}} c_{m-p}^{(2)} I_p^{(2)} + \frac{\mathsf{c}}{3}(m^2-m) c_m^{(2)}\,, \tag{4.3}$$

and the chiral primary states can be written as $|\Phi\rangle^{(2)} = |\xi\rangle^{(2)} \otimes c_1^{(2)}|0\rangle_{gh}$, where $|\xi\rangle^{(2)}$ is a primary state in the "matter + scalar" sector (for a spin-2 bc system $c_1|0\rangle_{gh}$ is the "true" ghost vacuum annihilated by all the positive modes b_n and c_n).

In the KM realization the generators read

$$\mathcal{L}_m^{(1)} = L_m^{(1)} + l_m^{(1)}, \quad l_m^{(1)} = \sum_{n\in\mathbf{Z}} n :b_{m-n}^{(1)} c_n^{(1)}: \tag{4.4}$$

$$\mathcal{H}_m^{(1)} = -\sum_{n\in\mathbf{Z}} :b_{m-n}^{(1)} c_n^{(1)}: + \sqrt{\frac{3-\mathsf{c}}{3}} I_m^{(1)}, \qquad \mathcal{Q}_m^{(1)} = b_m^{(1)}\,, \tag{4.5}$$

$$\mathcal{G}_m^{(1)} = 2\sum_{p\in\mathbf{Z}} c_{m-p}^{(1)} L_p^{(1)} + 2\sqrt{\frac{3-\mathsf{c}}{3}} \sum_{p\in\mathbf{Z}} (m-p) c_{m-p}^{(1)} I_p^{(1)} + \sum_{p,r\in\mathbf{Z}} (r-p) : b_{m-p-r}^{(1)} c_r^{(1)} c_p^{(1)} : + \frac{\mathsf{c}}{3}(m^2+m) c_m^{(1)} , \tag{4.6}$$

and the chiral primary states split as $|\Phi\rangle^{(1)} = |\xi\rangle^{(1)} \otimes |0\rangle_{gh}$.

Let us analyze the spectral flow mapping (2.6) at the level of these specific realizations; in other words, how the spectral flow maps the different DDK and KM fields into each other. Starting with the simplest transformation, $\mathcal{U}_1 \mathcal{G}_m^{(2)} \mathcal{U}_1^{-1} = \mathcal{Q}_m^{(1)}$, we get

$$\mathcal{U}_1 \, b_m^{(2)} \, \mathcal{U}_1^{-1} = b_m^{(1)} . \tag{4.7}$$

From the anticommutation relation $\{b_m, c_n\} = \delta_{m+n,0}$, for any bc system, and using (4.7), one obtains

$$\mathcal{U}_1 \, c_m^{(2)} \, \mathcal{U}_1^{-1} = c_m^{(1)} . \tag{4.8}$$

Now taking into account the ghost vacuum behaviour

$$b_{n\geq -1}^{(2)} |0\rangle_{gh} = c_{n\geq 2}^{(2)} |0\rangle_{gh} = 0, \qquad b_{n\geq 0}^{(1)} |0\rangle_{gh} = c_{n\geq 1}^{(1)} |0\rangle_{gh} = 0, \tag{4.9}$$

it is easy to deduce

$$\mathcal{U}_1 \sum_{n\in\mathbf{Z}} : b_{m-n}^{(2)} c_n^{(2)} : \mathcal{U}_1^{-1} = \sum_{n\in\mathbf{Z}} : b_{m-n}^{(1)} c_n^{(1)} : - \delta_{m,0} \tag{4.10}$$

and

$$\mathcal{U}_1 |0\rangle_{gh} = b_{-1}^{(1)} |0\rangle_{gh}, \qquad \mathcal{U}_{-1} |0\rangle_{gh} = c_1^{(2)} |0\rangle_{gh} . \tag{4.11}$$

From the transformations $\mathcal{U}_1 \mathcal{H}_m^{(2)} \mathcal{U}_1^{-1} = -\mathcal{H}_m^{(1)} - \frac{\mathsf{c}}{3}\delta_{m,0}$ and $\mathcal{U}_1 \mathcal{L}_m^{(2)} \mathcal{U}_1^{-1} = \mathcal{L}_m^{(1)} - m\mathcal{H}_m^{(1)}$, one finally obtains

$$\mathcal{U}_1 \, I_m^{(2)} \, \mathcal{U}_1^{-1} = I_m^{(1)} - \sqrt{\frac{3-\mathsf{c}}{3}} \delta_{m,0} \tag{4.12}$$

and

$$\mathcal{U}_1 \, L_m^{(2)} \, \mathcal{U}_1^{-1} = L_m^{(1)} - m\sqrt{\frac{3-\mathsf{c}}{3}} I_m^{(1)} + \delta_{m,0} . \tag{4.13}$$

The second of these expressions, although useful, does not contain any new information, since L_m contains the matter contribution (invariant under $\mathcal{U}_1$), plus the scalar contribution, whose spectral flow transformation is already given in the previous line. Thus we see, from (4.7), (4.8) and (4.12), that the spectral flow transformation, at the level of the DDK and KM realizations, does not mix fields of different nature, as it does at the topological algebra level (mapping $\mathcal{Q}_m$ and $\mathcal{G}_m$ into each other). The mirror map [6] does in fact mix the matter, the scalar and the ghost fields. This shows, again, a deep difference between the spectral flow map and the mirror map between the two twisted topological theories corresponding to a given N=2 superconformal theory.

To finish, let us consider the *reduced* DDK and KM conformal field theories (CFT's). These are the theories given by the "matter + scalar" systems, without the ghosts, and are described by the commutation relations

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{D}{12}(m^3-m)\delta_{m+n,0} \ , \\
[L_m, I_n] &= -nI_{m+n} - \tfrac{1}{2}Q_s(m^2+m)\delta_{m+n,0} \ , \\
[I_m, I_n] &= -m\delta_{m+n,0} \ .
\end{aligned} \tag{4.14}$$

where

$$L_m = L_m^{matter} - \frac{1}{2}\sum_n : I_{m-n}I_n : + \frac{1}{2}Q_s(m+1)I_m \tag{4.15}$$

and $D=26$ ($D=2$), $Q_s=\sqrt{\frac{25-d}{3}}$ ($Q_s=\sqrt{\frac{1-d}{3}}$) for the DDK (KM) realization.

Using the spectral flow transformations (4.12) and (4.13), between the DDK and KM CFT's, one readily deduces that primary fields, as well as null vectors, are mapped into each other. This was to be expected since the spectral flow maps topological $\mathcal{G}_0$-invariant null vectors into $\mathcal{Q}_0$-invariant null vectors and, at the level of the DDK and KM realizations, those reduce to ghost-free null vectors of the DDK and KM CFT's respectively [12].

5 Final Remarks

The main issue in this letter has been to write down and analyze the spectral flow mapping between the two twisted topological theories associated to a given N=2 superconformal theory. We have found that this mapping has better properties, at the level of descendant states, than the corresponding spectral flow on the untwisted N=2 superconformal theories: it preserves the level of the states and it maps null states into null states (spectral flows on the N=2 superconformal algebra do not map null states built on the chiral ring into null states built on the antichiral ring, and modify the level of the

states). Exactly for the same reasons this mapping has also a better behaviour than the mirror map between the two twisted theories.

We have also found that the spectral flow mapping interpolating between the two twisted theories gives rise to a topological algebra automorphism which acts inside a given theory. This is a reflection of the fact that for the N=2 superconformal theories the spectral flows commute with the twistings. Again, this automorphism preserves the level of the states, transforming null states into null states (the mirror map automorphism of the topological algebra fails to do that).

Both the spectral flow mapping and the automorphism, provide a powerful tool to compute topological null states, known to be related with Lian-Zuckermann states and other extra states relevant in string theory (see, for example [14]). As an example, we have written down all the level 2 results: the different types of general topological descendants with their spectral flow transformations and the specific coefficients which correspond to null states. We show that difficult null states can be computed straightforwardly from much easier ones.

We think, however, that the most interesting use of the spectral flow mapping written down here should be found at the level of the specific realizations of the topological algebra. We have just started this program, analyzing the behavior of the DDK and KM realizations of the topological algebra under the spectral flow action. We have found that, contrary to what happens with the mirror map, the spectral flow does not mix the different component fields (matter, scalar and ghosts). As a result, the null states of the reduced conformal field theories (without ghosts) are mapped into each other too.

In general, it may well happen that the image theory of a physically relevant theory is much simpler to deal with than the latter (like seems to be the case for the DDK and KM theories). In this respect let us remember that *almost all string theories, including the bosonic string, the superstring, and W-string theories, possess a twisted N=2 superconformal symmetry* [13]. There are several other interesting twisted N=2 topological theories, like the one considered in [14] for $d = 1$ string theory, or those related to the Kodaira-Spencer theory of gravity [15], or those considered in [6], etc (just to mention a few). It would be interesting to investigate whether the spectral flow mapping described here has something useful to add to the known results about all or some of those topological theories.

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